Game theory – A short introduction to utility and decisions

Leonardo Badia

leonardo.badia@gmail.com

Game theory

Introduction

Leonardo Badia - leonardo.badia@gmail.com

Game theory

- Game theory is the study of multiperson decision problems (=games)
  - Nothing to do with “games” as usually meant!!
- It was born as a branch of micro-economics and it is usually studied within this subject
- Many applications of game theory arise in the field of economics:
  - Micro level: trading, auctions, bargaining
  - Intermediate level: markets, firms
  - Macro level: countries, monetary authorities

Applied game theory

- More recently, researchers of many fields are studying game theory and its tools as related to their specific subjects of interest
  - Social sciences: mass behaviors, societal laws
  - Political sciences: elections, parties
  - Biology: behavior of herds, ecosystems
  - Computational intelligence: distributed thinking
  - Network systems: multi-agent algorithms

GT & Information Engineering

- System design and analysis (computational intelligence, algorithms, networks)
- GT captures the presence of multiple agents, which adhere to agreed protocols, pursue their own interests, interact with each other
  - Distributed systems, control protocols
  - Cooperation, coordination, synchronization
  - Problem solving (constrained optimization, distributed optimization)

GT & Information Engineering

- Moreover, game theory can be used for
  - Resource allocation and negotiation, fairness, manipulability, truthfulness, trust issues
  - Analysis of elections, electronic voting, trading systems, e-commerce, e-auctions
  - Shared and self-managed systems
  - Modal logics (common knowledge, beliefs)

Decision problems

First concept to test our know-how

Decision problem

- It consists of three elements:
  - **actions** belonging to a set A, which are what can be decided upon
  - **outcomes**, which are the results of said actions
  - **preferences**: a way to describe what outcomes are preferable for the decider

Preferences

- We have a set A of alternatives (at least two).
- A **preference** is a binary relationship $\succcurlyeq$ on A
  - If a, b $\in A$ and a $\succcurlyeq$ b then a is ranked above b
  - formally, it is reflexive and antisymmetric
- A preference $\succcurlyeq$ is said to be
  - **complete**, if $\forall$a,b$\in A$, then a $\succcurlyeq$ b or b $\succcurlyeq$ a (or both)
  - **transitive** if $\forall$a,b,c$\in A$, a $\succcurlyeq$ b and b $\succcurlyeq$ c $\Rightarrow$ a $\succcurlyeq$ c
- If $\succcurlyeq$ is complete+transitive, we call it **rational**
  - mathematician would call it a total order relation

Utility functions

- Utilities (also called payoff functions) are an arbitrary quantification $u(q)$ of the goodness coming from some input q
- If q is a countable good, $u(q)$ is generally an increasing function of q
  - and economists would say $u'(q) \geq 0, \quad u''(q) \leq 0$
- The exact formulation of u does not matter, it just maps the order via $\geq$ on numbers
- Preference relationship:
  - if $u(q) > u(q')$ a rational user prefers q over q'

Rationality

- The definition of utility hinders an important concept: players are rational
- This means that:
  - They act for their own good (selfishness)
  - They are aware of all consequences of their acts
- The first point may seem arguable
  - also it is inferred that rational is selfish!
- The most critical is actually the **second** one!

Rationality

- Many economists argue that human beings are far from being rational (they are often crazy or simply generous, or make mistakes)
  - These criticisms are irrelevant when dealing with computers, algorithms, autonomous agents
- The actual problem is the accuracy of the model, not rationality
  - We can simply modify the utility by accounting for generosity, illogical preferences, and so on

Preferences and utilities

- A preference $\succcurlyeq$ can be put in relationship with a **utility function** $u : A \rightarrow \mathbb{R}$.
- We say that u **represents** $\succcurlyeq$ if for all a, b $\in A$

$$a \succcurlyeq b \Leftrightarrow u(a) \geq u(b)$$

- **Theorem**: On a finite set A, $\succcurlyeq$ can be represented by u iff it is rational.

$\Rightarrow$Immediate (due to properties of $\geq$ over $\mathbb{R}$)

$\Leftarrow$ A suitable utility function can be

$$u\,(a) = |\,\{b \in A : a \succcurlyeq b\}\,|$$

Decision trees

- Setting a preference among alternatives is "easy" → just a maximization problem
- There can be subproblems:
  - Example 1: take the best route. It is the combination of multiple best routes (additive?)
  - Example 2: have the best meal. You can choose entree, main course, side, wine
- Strong assumption: one single smart agent can always solve such problems
- But what if you have multiple players?

Constitutions

Trying to unify preferences

Constitution

- Let $R(A)$ be a set of rational preferences on A
- A **constitution** (or **social welfare function**) is
$$f : R(A)^{\mathrm{n}} \rightarrow R(A)$$

- A constitution makes profile $\succcurlyeq_{(i)} = (\succcurlyeq_1, \succcurlyeq_2, \ldots \succcurlyeq_n)$ into a unique social preference $f(\succcurlyeq_{(i)})$

- Restricting preference $\succcurlyeq$ over A to $Y \subseteq A$:
$$\succcurlyeq|Y = \succcurlyeq \cap (Y \times Y)$$

Properties of constitutions

- A constitution f satisfies the Independence of Irrelevant Alternatives (**IIA**) if $\forall$ pairs of profiles $(\succcurlyeq_{(i)})$, $(\succcurlyeq'_{(i)})$ and $\forall$a, b $\in A$

$$\forall i, \succcurlyeq_i |\{a,b\} = \succcurlyeq'_i |\{a,b\}$$

$$\text{implies } f(\succcurlyeq_{(i)}) |\{a,b\} = f(\succcurlyeq'_{(i)}) |\{a,b\}$$

- that is, adding or removing elements to the alternative set does not change the relative priority order of a and b

Properties of constitutions

- Constitution f is **Pareto efficient** if $\forall$profiles $(\succcurlyeq_{(i)})$, $\forall$ a, b $\in A$
  $\forall i$, a $\succcurlyeq_i$ b implies a $\succcurlyeq$ b , where $\succcurlyeq\, = f(\succcurlyeq_{(i)})$
  - that is, if everybody prefers a over b, so does the society as a whole as dictated by the social rule

- Pareto efficiency relates to the concept of "being better for everybody"

Properties of constitutions

- f is a **dictatorship** if there exists i such that

  $$a \succcurlyeq_i b \quad \text{implies} \quad a \succcurlyeq b\text{, where } \succcurlyeq \, = f(\succcurlyeq_{(i)})$$

  - i.e., the constitution simply mimics i 's preference

- f is **monotonic** if, when a single individual modifies his/her preference ranking something better, f does not rank it worse
- f satisfies **non-imposition** if all rational preferences can be outputs, i.e., is surjective

Arrow's Theorem

- Theorem (Arrow, 1951).
- Impossible to design a constitution which is:
  - non-dictatorship
  - monotonic
  - satisfies IIA and non-imposition

- A more synthetic version (1963) says that if f
  - is Pareto efficient
  - satisfies IIA

  …then it is a dictatorship!

Elections and Paradoxes

which do not hold only for elections

Elections and democracy

- What is **democracy**?
- Usually we immediately connect democracy with elections, as well as with "majority rule"
- What does majority means?
- Things get complicated in the case of multiple choices

Elections and democracy

- Say we have 3 voters and 2 candidates
- The preference are as follows

voter	1	2	3
best	A	A	B
worst	B	B	A

- A beats B by majority rule since 2 people prefer A over B and only 1 does the opposite
- A democratic society should choose A

Elections and democracy

- Say we have 3 voters and 3 candidates
- The preference are as follows

voter	1	2	3
best	A	A	B
	B	C	C
worst	C	B	A

- A>B, B>C, A>C. A beats all other candidates
- A democratic society should choose A

Elections and democracy

- Say we have 3 voters and 3 candidates
- The preference are as follows

voter	1	2	3
best	A	C	B
	B	A	C
worst	C	B	A

- A>B, B>C, C>A. There is no "best" candidate.
- What should a democratic society choose?

Cycle → Paradox!

Nicolas de Condorcet

- 1743-1794
- French mathematician, economist, politician
- Representative of the "moderate side" during the French revolution

Terminology

- A candidate that beats majority-wise all the others is called the **Condorcet winner**
- If there is no winner, then there must be a cycle, formally called a **Condorcet cycle**
- Also mixed cases are possible for >3 candidates (e.g., a winner, and a cycle among the remaining 3)

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A		
	B	A	B
		B	
worst			A

- It all depends where we put C between A and B

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A	C	C
	B	A	B
	C	B	
worst			A

- In this case, C is the Condorcet winner

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A		
	B	A	B
	C	B	C
worst		C	A

- C is the worst of all ("Condorcet loser")

Remark 1

- The cases with three candidates directly originate from the case with two

voter	1	2	3
best	A	C	
	B	A	B
	C	B	C
worst			A

- Condorcet cycle!

Remark 2

- Condorcet cycles cannot occur when only two alternatives are present
- With ≥ 3 alternatives there may be cycles
- The probability of Condorcet cycles grows with the number of candidates
- If preferences are sufficiently randomized, for large ($\to \infty$) number of candidates, Condorcet cycles are sure to occur

Remark 2

- Probability of at least one cycle (random preferences)

voters→ choices↓	3	5	7	9	∞
3	5.6%	6.9%	7.5%	7.8%	8.8%
5	16.0%	20.0%	21.5%	23.0%	25.1%
7	23.9%	29.9%	30.5%	34.2%	36.9%
∞	100.0%	100.0%	100.0%	100.0%	100.0%

Remark 3

- Even though we speak of candidate and elections, the same thing could apply to:
- Scheduling: think of candidate A, B, C, as users/ packets/ objects to allocate and voters 1, 2, 3, as criteria to choose among them
- Optimization: think of candidate A, B, C, as possible solutions to an optimization problem and voters 1, 2, 3, as possible goal functions

Some "real world" examples

- Fiscal politics of governments

	liberals	anti-deficit	conservatives
best	Taxes ↑ Spending ↑	Taxes ↑ Spending ↓	Taxes ↓ Spending ↓
	Taxes ↓ Spending ↓	Taxes ↑ Spending ↑	Taxes ↑ Spending ↓
worst	Taxes ↑ Spending ↓	Taxes ↓ Spending ↓	Taxes ↑ Spending ↑

Some "real world" examples

- Quality of Service

	"well behaved"	high delay	high losses
best	Voice over IP	Video Streaming	Best Effort Data
	Video Streaming	Best Effort Data	Voice over IP
worst	Best Effort Data	Voice over IP	Video Streaming

Search for a perfect system

does it exist, actually?

Setting the agenda

- Assume 3 competitors A, B, and C: we choose between A and B in a first round, then the winner goes up against C
- Seems fair? It is not in a Condorcet cycle!
- Assume the cycle is A<B<C<A: C wins, while he would lose in a different setup
- For example: choose between C and B first, then the winner goes up against A: A wins

Other methods

- There are actually many electoral systems (which work also as selection rules in allocation problems), such as
- Plurality voting
- Two-phase Run-off
- Borda counting
- Approval voting
- Instant run-off

Plurality voting

- Let each voter sort the candidates in order of personal preference
  - Some candidates will get "first place" by some voters
- In the "plurality voting" criterion, the winner is who has most first places among the voters
- Is this mechanism immune to paradoxes?

Plurality voting

- Assume we have 9 voters

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	C
	B	C	B
worst	C	A	A

- A wins (4 votes vs. 3 votes of B and 2 of C)
- However a majority prefers B>A
- A majority also prefers C>A
- There even is a Condorcet winner (B), as B>C

Two-phase Run-off

- We make a two-round voting
- First we select the two best candidates
- In a second round, we choose between them in a ballot

Two-phase Run-off

- Again, assume we have 9 voters

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	C
	C	C	B
worst	B	A	A

- A and B go to the ballot, B wins 5-4
- However a majority prefers C>A and C>B
- C is the Condorcet winner, but C does not even make it to the ballot

Borda count

- Plurality and Run-off favor "polarized" solutions over "compromise" solutions
- A strong candidate in a (large) minority wins over a weak one even if appreciated by many
- Borda count tries to solve this:
  - If we have M candidates, the voter gives a score
  - M-1 points go to the best one, M-2 to the next one and so on; the last one gets 0 points
- Is this method better?

Borda count

- We have again 9 voters (assigning 27 points)

	1-5 (5 voters)	6-8 (3 voters)	9
best	A	B	C
	B	C	B
worst	C	A	A

- A achieves 10 points, B 12, C 5. B wins
- However, A is the Condorcet winner, since A>B and A>C
- Similar paradoxes hold for different scores

Borda count with dropout

- Borda-like counts are used, e.g., for sports

	1-5 (5 voters)	6-7 (2 voters)	8-9 (2 voters)
best	D	A	A
	C	D	B
	B	B	D
worst	A	C	C

- Total points: A 12, B 11, C 10, D 21
- Thus: D gold, A silver, B bronze

Borda count with dropout

- But D retires (e.g. anti-doping or naked photo)

	1-5 (5 voters)	6-7 (2 voters)	8-9 (2 voters)
best		A	A
	C		B
	B	B	
worst	A	C	C

- Total points: A 8, B 9, C 10
- Thus: C gold, B silver, A bronze
- The retirement entirely reverse the order

Approval voting

- Each voter can give more than one preference
- Each preference assigns one point
- The number N of preferences must be between 1 and M (no. of candidates)
- For N=1 we fall back into plurality case

Approval voting

- Again, an example with the 9 voters

	1-3 (3 voters)	4-6 (3 voters)	7-8 (2 voters)	9
best	A	D	B	A
	C	B	D	B
	D	C	C	C
worst	B	A	A	D

- Top 2 approvals: A 4, B 6, C 3, D 5. B wins
- Top 3 approvals: A 4, B 6, C 9, D 8. C wins
- The result depends on N

Approval voting

- Every system has a different outcome.

	1-3 (3 voters)	4-6 (3 voters)	7-8 (2 voters)	9
best	A	D	B	A
	C	B	D	B
	D	C	C	C
worst	B	A	A	D

- Plurality -Top 1 approvals- prefers A (4 votes)
- Borda winner is D with 16 (A 12, B 14, C 12)

Instant Run-off

- Again, we ask each voter for its "order of preference"
- Only top preferences count to reach a majority
- We make ("instantaneously") subsequent rounds, each time removing the candidate with least top preferences

Instant Run-off

- Let see an example with 17 voters

	6 voters	5 voters	4 voters	2 voters
best	A	C	B	B
	B	A	C	A
worst	C	B	A	C

- No majority, so candidate C is eliminated
- A gains 5 votes, and wins with 11 votes
- It seems logical (A is the Condorcet winner)

Instant Run-off

- What if the last 2 voters chose A first instead of B

	6 voters	5 voters	4 voters	2 voters
best	A	C	B	A
	B	A	C	B
worst	C	B	A	C

- This causes A to lose! B is now eliminated at the first round. 4 votes go to C, who wins with 9 votes
- A loses due to an increasing consensus

Setting the agenda

- The selection of a particular method may advantage some competitors in an almost invisible way
- This is a very subtle factor in many fields: politics, sports, sciences, everyday life

- Fortunately, this power is not almighty

Setting the agenda

- A>B>C>A are in Condorcet cycle. D is worst.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	C	B
	B	A	C
	C	B	A
worst	D	D	D

- There is no way for D to win (A>D, B>D, C>D)
- However, if we make semifinals and final, it always win who goes against D first

Setting the agenda

- A>B>C>A are in Condorcet cycle. D is best.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	D	D	D
	A	C	B
	B	A	C
worst	C	B	A

- Here, D always wins and the order of A, B, C depends on the agenda setting

Cheating: Condorcet cycles

another consequence of this paradox

Cheating

- A>B>C>A are in a Condorcet cycle.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	C
	D	A	B
	C	D	D
worst	B	C	A

- However, A is the winner in many systems (plurality, Borda count, Top 2 approval…)
- Assume we choose plurality: A wins

Cheating

- 8 and 9 are disappointed. For them A is worst

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	~~C~~ B
	D	A	~~B~~ C
	C	D	D
worst	B	C	A

- They decide to cheat and indicate B as preferred choice, instead of C.
- Now B wins. For them it is an improvement.

Cheating

- For the first 4 voters this is bad.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	B	~~C~~ B
	D	A	~~B~~ C
	C	D	D
worst	B	C	A

- They may protest and ask for help from 5-7, but these are happy, since B is best for them

Cheating

- But if they can act first, they can cheat too

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	~~A~~ C	B	C
	~~D~~ A	A	B
	~~C~~ D	D	D
worst	B	C	A

- It counteracts cheating by 8-9, who vote C
- Bad for 5-7 but they can't do anything
- C wins with only 2 "natural" votes

Cheating

- There is also a chance for B's supporters.

	1-4 (4 voters)	5-7 (3 voters)	8-9 (2 voters)
best	A	~~B~~ A	C
	D	~~A~~ B	B
	C	D	D
worst	B	C	A

- They can change and support A (whom they prefer better than C): now A wins again…

…in the end it all depends on who *cheat first*

Extensions to Arrow's Theorem

- Social function f is **strategy-proof** (non-manipulable) if for any profile $(\succcurlyeq_{(i)})$ and a certain preference $\succcurlyeq'_j$

$$f(\succcurlyeq_{(i)}) \succcurlyeq_j f(\succcurlyeq'_j, \succcurlyeq_{-j})$$

  - that is, no one has incentive to cheating

- **Gibberard-Satterthwaite theorem**. Any strategy-proof constitution that does not forbid anyone to win… must be a dictatorship!

Problems of electoral systems

- It seems that no good system exists
- Recall Arrow's Theorem if a constitution:
  - is Pareto efficient
  - satisfies IIA

  …then it is a dictatorship!
- "Ways out"
  - some conditions are weakened
  - use free approval voting (vote "for" or "against")
  - we restrict the profile

Majority rule

- This last solution has been proposed in various ways by many economists and is in short a way to apply **majority rule**
- Formally, majority rule $\succcurlyeq$ can be defined as:

$$a \succcurlyeq b \quad \Leftrightarrow \quad |\{i : a \succcurlyeq_i b\}| \geq |\{i : b \succcurlyeq_i a\}|$$

  - is Pareto efficient
  - satisfies IIA
  - is not a dictatorship

…but is not a constitution!

Majority rule

- Majority rule is complete but **non-transitive**
- The reason is… Condorcet cycles!
- If we are able to eliminate Condorcet cycles, majority rule becomes a constitution and possesses "nice" properties (Sen)
- Alternative: focus only on cases with a *linear* order relationship on set A
  - This also guarantees to avoid Arrow's theorem by using majority rule (Black)